Indian Statistical Institute, Bangalore Centre.
Mid-Semester Exam : Graph Theory

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Max. points : 20. **Time Limit : 1.5 hours.**

There are two parts to the question paper - PART A and PART B. Read the instructions for each section carefully. Some common notations and definitions are listed on Page 3.

1 PART A : MULTIPLE-CHOICE QUESTIONS - 10 Points.

Please write only the correct choice(s) (for ex., (a), (b) et al.) in your answer scripts. No explanations are needed. Write PART A answers in a separate page.

Some questions will have multiple correct choices. Answer all questions. Each question carries 2 points.

1. What is the diameter of the hypercube graph $\{0,1\}^n$?
   (a) $n/2$
   (b) $2n$
   (c) $n-1$
   (d) n

2. How many labelled spanning trees does the cycle graph on n vertices have ?
   (a) n
   (b) $\lfloor n/2 \rfloor$
   (c) $n-1$
   (d) $\lceil n/2 \rceil$

3. What is the matching number (i.e., α') of the Petersen graph ?
   (a) 3
   (b) 4
   (c) 5
   (d) 6
   (e) 7
4. Which of the following graphs are bi-partite ?
   (a) Complete graph $K_n, n \geq 3$.
   (b) Petersen graph.
   (c) Hypercube graph on $\{0, 1\}^n$.
   (d) Path graph on n vertices.
5. How many distinct 'independent sets' does the complete graph K_n have ?
   (a) 1
   (b) $\lfloor n/2 \rfloor$
   (c) $n - 1$
   (d) n.

2 PART B : 10 Points.

Answer any two questions only. All questions carry 5 points.

Give necessary justifications and explanations for all your arguments. If you are citing results from the class, mention it clearly. Always define the underlying random variables and events clearly before computing anything !

1. Suppose T, T' are spanning trees of G and $e \in T - T'$. Then $\exists e' \in T'$ such that $T - e + e'$ is also a spanning tree.
2. Show that a simple graph on n vertices with $\lfloor n^2/4 \rfloor$ edges and no triangles, then it is the complete bipartite graph $K_{k,k}$ if $n = 2k$ or $K_{k,k+1}$ if $n = 2k + 1$.
3. For $k \geq 0$, show that a k-regular bipartite graph has a perfect matching.

Some notations :

- G is assumed to be a finite simple graph everywhere.
- d_G is defined as the usual graph metric when all edge weights are taken to be 1 and $diam(G) := \max\{d_G(u,v) : u, v \in V(G)\}$.
- Q_n is the hypercube graph on $\{0,1\}^n$ i.e., $V = \{0,1\}^n$ and $x \sim y$ is x and y differ exactly in one-coordinate i.e., $|\{i : x_i \neq y_i\}| = 1$ where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.
- $\alpha'(G)$ - Maximum independent edge set ; $\beta'(G)$ - Minimum edge cover.
- $\alpha(G)$ - Maximum independent vertex set ; $\beta(G)$ - Minimum vertex cover.